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On the measure of conflicts: Shapley Inconsistency Values ☆

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ABSTRACT

There are relatively few proposals for inconsistency measures for propositional belief bases. However inconsistency measures are potentially as important as information measures for artificial intelligence, and more generally for computer science. In particular, they can be useful to define various operators for belief revision, belief merging, and negotiation. The measures that have been proposed so far can be split into two classes. The first class of measures takes into account the number of formulae required to produce an inconsistency: the more formulae required to produce an inconsistency, the less inconsistent the base. The second class takes into account the proportion of the language that is affected by the inconsistency: the more propositional variables affected, the more inconsistent the base. Both approaches are sensible, but there is no proposal for combining them. We address this need in this paper: our proposal takes into account both the number of variables affected by the inconsistency and the distribution of the inconsistency among the formulae of the base. Our idea is to use existing inconsistency measures in order to define a game in coalitional form, and then to use the Shapley value to obtain an inconsistency measure that indicates the responsibility/contribution of each formula to the overall inconsistency in the base. This allows us to provide a more reliable image of the belief base and of the inconsistency in it.

1. Introduction

There are numerous works on reasoning under inconsistency. One can quote for example paraconsistent logics, argumentation frameworks, belief revision and fusion, etc. All these approaches illustrate the fact that the dichotomy between consistent and inconsistent sets of formulae that comes from classical logics is not sufficient for describing these sets. As shown by these works, normally when given two inconsistent sets of formulae, they are not trivially equivalent. They do not contain the same information and they do not contain the same contradictions.

Measures of information *à la* Shannon have been studied in logical frameworks (see for example [31]). Roughly they involve counting the number of models of the set of formulae (the less models, the more informative the set). The problem is that these measures regard an inconsistent set of formulae as having a null information content, which is counter-intuitive (especially given all the proposals for paraconsistent reasoning). So generalizations of measures of information have been proposed to solve this problem [39,53,36,32,24].

In comparison, there are relatively few proposals for inconsistency measures [22,27,35,32,28,18]. However, these measures are potentially important in diverse applications in artificial intelligence, such as belief revision, belief merging, and negotiation, and more generally in computer science. Already some provisional studies indicate that measuring inconsistency

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may be seen to be a useful tool in analysing a diverse range of information types including news reports [29], integrity constraints [18], software specifications [9,10,42], and ecommerce protocols [12].

The current proposals for measuring inconsistency can be classified in two approaches. The first approach involves "counting" the minimal number of formulae needed to produce the inconsistency. The more formulae needed to produce the inconsistency, the less inconsistent the set [35]. This idea is an interesting one, but it rejects the possibility of a more fine-grained inspection of the (content of the) formulae. In particular, if one looks to singleton sets only, one is back to the initial problem, with only two values: consistent or inconsistent.

The second approach involves looking at the proportion of the language that is touched by the inconsistency. This allows us to look *inside* the formulae [27,32,18]. This means that two formulae viewed as two whole belief bases (singleton sets) can have different inconsistency measures. But, in these approaches one can identify the set of formulae with its conjunction (i.e. the set $\{\varphi, \varphi'\}$ has the same inconsistency measure as the set $\{\varphi \wedge \varphi'\}$). This can be sensible in some applications, but this means that the distribution of the contradiction among the formulae is not taken into account.

What we propose in this paper is a definition for inconsistency measures that allow us to take the best of the two approaches. This will allow us to build inconsistency measures that are able to look inside the formulae, but also to take into account the distribution of the contradiction among the different formulae of the set.

The above-mentioned approaches define *inconsistency measures*, i.e. functions that associate a number to each belief base. These global base-level measures are sufficient for a variety of applications. But in some cases we need an evaluation on a finer level, that is for each formula of the base. We call these functions, that associate a number to each formula of a base, *inconsistency values*. Such a function allows us to identify which are the most problematic formulae of a belief base with respect to the inconsistency. This can be very useful for applications such as belief revision or negotiation. These inconsistency values provide a more detailed view of the inconsistency, and they can be used to defined new inconsistency measures which more accurately reflect the inconsistency of the whole base.

To this end we will use a notion that comes from coalitional game theory: the Shapley value. This value assigns to each player the payoff that this player can expect from her utility for each possible coalition. The idea is to use existing inconsistency measures (that allow us to look inside the formulae) in order to define a game in coalitional form, and then to use the Shapley value to obtain an inconsistency measure with the desired properties. From these inconsistency values, it is possible to define new interesting inconsistency measures. We present these measures, we state a set of logical properties they satisfy, and we show that they are more interesting than the other existing measures.

The plan of the paper is as follows: After some preliminaries in the next section, Section 3 introduces inconsistency measures that count the number of formulae needed to produce an inconsistency. Section 4 presents the approaches where the inconsistency measure depends on the number of variables touched by the inconsistency. Section 5 introduces the problem studied in this paper and illustrates that the naive solution is not adequate. Section 6 gives the definition of coalitional games and of the Shapley value. Section 7 introduces the inconsistency measures based on Shapley value. Then we study the logical properties of these measures in Section 8, and we provide a complete axiomatization of a particular measure in Section 9 through a set of intuitive axioms. Section 10 sketches the possible applications of those measures for reasoning and for belief change operators. Finally Section 11 concludes by giving perspectives of this work and its possible applications for belief change operators.

2. Preliminaries

We will consider a propositional language $\mathcal{L}$ built from a finite set of propositional symbols $\mathcal{P}$. We will use $a, b, c, \ldots$ to denote the propositional variables, and Greek letters $\alpha, \beta, \varphi, \ldots$ to denote the formulae. An interpretation is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula φ, denoted $\omega \models \varphi$, if and only if it makes φ true in the usual truth-functional way. $Mod(\varphi)$ denotes the set of models of the formula φ, i.e. $Mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$. We will use $\subseteq$ to denote the set inclusion, and we will use $\subset$ to denote the strict set inclusion, i.e. $A \subset B$ iff $A \subseteq B$ and $B \nsubseteq A$. Let A and B be two subsets of C, we note $C = A \oplus B$ if A and B form a partition of C, i.e. $C = A \oplus B$ iff $C = A \cup B$ and $A \cap B = \emptyset$. We will denote the set of real numbers by $\mathbb{R}$.

A *belief base* K is a finite set of propositional formulae. More exactly, as we will need to identify the different formulae of a belief base in order to associate them with their inconsistency value, we will consider belief bases K as vectors of formulae. For logical properties we will need to use the set corresponding to each vector, so we suppose that there is a function mapping each vector $K = (\alpha_1, \ldots, \alpha_n)$ into $\bar{K}$, the set $\{\alpha_1, \ldots, \alpha_n\}$. As it will never be ambigous, in the following we will omit the graphical distinction and write K as both the vector and the set.

Let us note $\mathcal{K}_{\mathcal{L}}$ the set of belief bases definable from formulae of the language $\mathcal{L}$. A belief base is consistent if there is at least one interpretation that satisfies all its formulae.

If a belief base K is not consistent, then one can define the minimal inconsistent subsets of K as:

$$\mathsf{MI}(K) = \left\{K' \subseteq K \;\middle|\; K' \vdash \bot \text{ and } \forall K'' \subset K', K'' \nvdash \bot\right\}$$

If one wants to recover consistency from an inconsistent base K by removing some formulae, then the minimal inconsistent subsets can be considered as the purest form of inconsistency. To recover consistency, one has to remove at least one formula from each minimal inconsistent subset [49]. The notion of maximal consistent subset is the dual of that of minimal inconsistent subset. Each maximal consistent subset represents a maximal (by set inclusion) subset of the base that is consistent.

$$\mathsf{MC}(K) = \{K' \subseteq K \mid K' \nvdash \perp \text{ and } \forall K'' \text{ s.t. } K' \subset K'', K'' \vdash \perp\}$$

A *free formula* of a belief base K is a formula of K that does not belong to any minimal inconsistent subset of the belief base K, or equivalently a formula that belongs to every maximal consistent subset of the belief base. This means that this formula has nothing to do with the conflicts of the base.

3. Inconsistency measures based on formulae

When a base is inconsistent the classical inference relation is trivialized, since one can deduce every formula of the language from the base (*ex falso quodlibet*). To address this problem, paraconsistent reasoning techniques have been developed to only allow non-trivial consequences to follow from an inconsistent base. There is a range of paraconsistent systems, each based on a weakening of classical reasoning. One approach is a very straightforward weakening that only allows inferences from consistent subsets of a base rather than from the whole base (e.g. [43,4,45]).

Paraconsistent reasoning systems provide a natural starting point for analysing inconsistency. One interesting option is to analyse inconsistency in terms of the maximal consistent subsets of the base. So one can use for instance the size (or the number) of those maximal consistent subsets as a measure of the inconsistency. Indeed analysis of the maximal consistent subsets of a base is the basis of the measure of inconsistency proposed by Knight [35,36] which we review next.

Definition 1. A *probability function* on $\mathcal{L}$ is a function $P : \mathcal{L} \to [0, 1]$ s.t.:

- if $\models \alpha$, then $P(\alpha) = 1$.
- if $\models \neg(\alpha \wedge \beta)$, then $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$.

See [46] for more details on this definition. In the finite case, this definition gives a probability distribution on the interpretations, and the probability of a formula is the sum of the probability of its models.

Then the inconsistency measure defined by Knight [35] is given by:

Definition 2. Let K be a belief base.

- K is *η-consistent* ($0 \leqslant \eta \leqslant 1$) if there is a probability function P such that $P(\alpha) \geqslant \eta$ for all $\alpha \in K$.
- K is *maximally η-consistent* if η is maximal (i.e. if $\gamma > \eta$ then K is not γ-consistent).

As is easily seen, in the finite case, a belief base is maximally 0-consistent if and only if it contains a contradictory formula. And a belief base is maximally 1-consistent if and only if it is consistent.

The notion of *maximal η-consistency* can be used as an inconsistency measure. This is the direct formulation of the idea that the more formulae are needed to produce the inconsistency, the less this inconsistency is problematic. Let us illustrate this on the following “lottery example”:

Example 1. There are a number of lottery tickets with one of them being the winning ticket. Suppose w_i denotes ticket i will win, then we have the assumption $w_1 \vee \cdots \vee w_n$. In addition, for each ticket i, we may pessimistically (or probabilistically if the number of tickets is important) assume that it will not win, and this is represented by the assumption $\neg w_i$. So the base K_L is:

$$K_L = \{\neg w_1, \ldots, \neg w_n, w_1 \vee \cdots \vee w_n\}$$

Clearly if there are three or two (or one!) tickets in the lottery, then this base is highly inconsistent. But if there are millions of tickets there is intuitively (nearly) no conflict in the base. This is expressed by the η-consistency measure, since the base K_L is maximally $(n-1)/n$-consistent.[1] So with three tickets the base is maximally 2/3-consistent, and with a million tickets we are very close to maximally 1-consistent.

Example 2. Let $K_1 = \{a, b, \neg a \vee \neg b\}$. Since, we can reflect a distribution over models by $P(a \wedge b) = \frac{1}{3}$, $P(a \wedge \neg b) = \frac{1}{3}$, and $P(\neg a \wedge b) = \frac{1}{3}$. We get $P(a) = \frac{2}{3}$, $P(b) = \frac{2}{3}$, and $P(\neg a \vee \neg b) = \frac{2}{3}$. As a result, K_1 is maximally $\frac{2}{3}$-consistent.

Let $K_2 = \{a \wedge b, \neg a \wedge \neg b, a \wedge \neg b\}$. K_2 is maximally $\frac{1}{3}$-consistent, whereas each subbase of cardinality 2 is maximally $\frac{1}{2}$-consistent.

[1] Unless stated otherwise, we consider in the examples that the set of propositional symbols $\mathcal{P}$ of the language is exactly the set of propositional symbols that appear in the base K.

For minimal inconsistent sets of formulae, computing this inconsistency measure is easy:

Proposition 1. *(See [35].) If $K' \in \mathsf{MI}(K)$, then K' is maximally $\frac{|K'|-1}{|K'|}$-consistent.*

But in general this measure is harder to compute than the case considered above. However it is possible to compute it using the simplex method [35].

4. Inconsistency measures based on variables

Another method to evaluate the inconsistency of a belief base is to look at the proportion of the language concerned with the inconsistency. To this end, it is clearly not possible to use classical logics, since the inconsistency contaminates the whole language. But if we look at the two bases $K_3 = \{a \wedge \neg a \wedge b \wedge c \wedge d\}$ and $K_4 = \{a \wedge \neg a \wedge b \wedge \neg b \wedge c \wedge \neg c \wedge d \wedge \neg d\}$, we can observe that in K_3 the inconsistency is mainly about the variable a, whereas in K_4 all the variables are touched by a contradiction. This is the kind of distinction that these approaches allow.

One way to circumscribe the inconsistency only to the variables directly concerned is to use multi-valued logics, and especially three-valued logics, with the third "truth value" denoting the fact that there is a conflict on the truth value (true–false) of the variable.

We do not have space here to detail the range of different measures that have been proposed. See [22,27,32,24,18] for more details on these approaches. We only give one such measure, that is a special case of the degrees of contradiction defined in [32]. The idea of the definition of these degrees in [32] is, given a set of tests on the truth value of some formulae of the language (typically on the variables), the degree of contradiction is the cost of a minimum test plan that ensures recovery of consistency.

The inconsistency measure we define here is the (normalized) minimum number of inconsistent truth values in the LP_m models [48] of the belief base. Let us first introduce the LP_m consequence relation.

- An interpretation ω for LP_m maps each propositional atom to one of the three "truth values" $\mathsf{F}, \mathsf{B}, \mathsf{T}$, the third truth value B meaning intuitively "both true and false". $3^{\mathcal{P}}$ is the set of all interpretations for LP_m. "Truth values" are ordered as follows: $\mathsf{F} <_t \mathsf{B} <_t \mathsf{T}$. The interpretations are extended to formulae as follows:

  - $\omega(\top) = \mathsf{T}$, $\omega(\bot) = \mathsf{F}$
  - $\omega(\neg\alpha) = \mathsf{B}$ iff $\omega(\alpha) = \mathsf{B}$
  - $\omega(\neg\alpha) = \mathsf{T}$ iff $\omega(\alpha) = \mathsf{F}$
  - $\omega(\alpha \wedge \beta) = \min_{\leqslant_t}(\omega(\alpha), \omega(\beta))$
  - $\omega(\alpha \vee \beta) = \max_{\leqslant_t}(\omega(\alpha), \omega(\beta))$

- The set of models of a formula φ is:

  $$Mod_{LP}(\varphi) = \{\omega \in 3^{\mathcal{P}} \mid \omega(\varphi) \in \{\mathsf{T}, \mathsf{B}\}\}$$

  Define $\omega!$ as the set of "inconsistent" variables in an interpretation ω, i.e.

  $$\omega! = \{x \in \mathcal{P} \mid \omega(x) = \mathsf{B}\}$$

  Then the minimal models of a formula are the "most classical" ones (i.e. the models with the largest subset by set inclusion of atoms assigned either T or F):

  $$\min\bigl(Mod_{LP}(\varphi)\bigr) = \{\omega \in Mod_{LP}(\varphi) \mid \nexists \omega' \in Mod_{LP}(\varphi) \text{ s.t. } \omega'! \subset \omega!\}$$

  The LP_m consequence relation is then defined by:

  $$K \models_{LP_m} \varphi \quad \text{iff} \quad \min\bigl(Mod_{LP}(K)\bigr) \subseteq Mod_{LP}(\varphi)$$

So φ is a consequence of K if all the "most classical" models of K are models of φ.

Then let us define the LP_m measure of inconsistency [32], noted I_{LP_m}, as:

Definition 3. Let K be a belief base. $I_{LP_m}(K) = \frac{\min_{\omega \in Mod_{LP}(K)}\{|\omega!|\}}{|\mathcal{P}|}$.

So, with I_{LP_m}, the measure of inconsistency of a base is basically the number of "inconsistent" variables in the most classical models.

Example 3. $K_5 = \{a \wedge \neg a, b, \neg b, c\}$. The "most classical" LP model of K_5 is ω with $\omega(a) = \mathsf{B}$, $\omega(b) = \mathsf{B}$, $\omega(c) = \mathsf{T}$. So with 2 of the 3 variables being B, this gives $I_{LP_m}(K_5) = \frac{2}{3}$.

In this example one can see the advantage in these kinds of measures compared to measures based on formulae since this base is maximally 0-consistent because of the contradictory formula $a \wedge \neg a$. But there are also non-trivial formulae in the base, and this base is not totally inconsistent according to I_{LP_m}.

Conversely, measures based on variables like this one are unable to take into account the distribution of the contradiction among formulae. In fact the result would be exactly the same with $K_5' = \{a \wedge \neg a \wedge b \wedge \neg b \wedge c\}$. This can be sensible in some applications, but in some cases this can also be seen as a drawback. In particular when the formulae represent different pieces of information (that can come from different sources for instance). See [34] for a related discussion.

5. Dimensions for measuring inconsistency

From the discussion in the previous sections, we can regard the measures of propositional inconsistency (so far) as falling into one or other of the following two classes.

Formula-centric measures These measures take into account the number of formulae required for inconsistency: Fewer formulae means higher degree of inconsistency. This is exemplified by the notion of η-consistent.

Atom-centric measures These measures take into account the proportion of the language affected by inconsistency: More propositional variables involved in inconsistency means higher degree of inconsistency. This is exemplified by the I_{LP_m} measure of inconsistency.

We can also note that the measures defined in the previous sections provide a measure of inconsistency for the whole base. But we can consider another type of measure, and this leads us to the following two classes of measures.

Inconsistency measures (Base-level measures) These measures provide a measure of inconsistency to the beliefbase as a whole. They do not assign a measure of inconsistency to individual formulae.

Inconsistency values (Formula-level measures) These values provide a measure of inconsistency to the formulae in a beliefbase, and in a sense assign the degree of blame or responsibility that can be ascribed to the formulae for the inconsistencies arising in the beliefbase.

The measures reviewed up to now are all inconsistency measures (base-level measures). The aim of this paper is to present and investigate inconsistency values (formula-level measures), and to show that they can be very useful for a lot of reasoning applications such as belief revision, inference, negotiation, etc.

In order to explore the notion of an inconsistency value, we first consider a simple option for such a value.

Definition 4. For an inconsistency measure on beliefbases, $I : \wp(\mathcal{L}) \mapsto [0,1]$, a base $K \subseteq \mathcal{L}$ and $\alpha \in K$, the *simple assignment* to formulae, $M_I^K : \mathcal{L} \mapsto [0,1]$, is as follows,

$$M_I^K(\alpha) = I(K) - I\big(K \setminus \{\alpha\}\big)$$

Let us see an example when we take as inconsistency measure the simplest possible option:

Definition 5. The *drastic inconsistency value* is defined as:

$$I_d(K) = \begin{cases} 0 & \text{if } K \text{ is consistent} \\ 1 & \text{otherwise} \end{cases}$$

Example 4. For this example, let us use the drastic inconsistency value I_d, and consider $K_6 = \{a, a \wedge b, \neg a\}$. Hence, we get the following.

$$M_{I_d}^K(a) = 0, \qquad M_{I_d}^K(a \wedge b) = 0, \qquad M_{I_d}^K(\neg a) = 1$$

In this example, we see that all the blame is assigned to the formula that is in both minimal inconsistent subsets, and none is assigned to either of the formulae which appear in only one of the minimal inconsistent subsets. This is clearly a significant shortcoming with the simple assignment since at least some of the blame should be assigned to a and to $a \wedge b$, and hence they should each have a non-zero evaluation.

We see in the following example that by taking a more refined inconsistency measure on beliefbases, we get a better assignment to formulae.

Definition 6. The *MI inconsistency measure* is defined as the number of minimal inconsistent sets of K, i.e.:

$$I_{MI}(K) = \big|\mathsf{MI}(K)\big|$$

Example 5. $K_6 = \{a, a \wedge b, \neg a\}$. Hence, we get the following.

$$M^K_{I_{MI}}(a) = 1, \qquad M^K_{I_{MI}}(a \wedge b) = 1, \qquad M^K_{I_{MI}}(\neg a) = 2$$

However, even with this more refined inconsistency measure on beliefbases, we still can find problematical examples, such as the "lottery example" (Example 1). Let us see what this example gives here:

Example 6. For this example, let $I_{MI}(K) = |\mathsf{MI}(K)|$ and let

$$K_L = \{\neg w_1, \ldots, \neg w_n, w_1 \vee \cdots \vee w_n\}$$

Hence, for each $\alpha \in K_L$, we get $M^K_{I_{MI}}(\alpha) = 1$.

The problem with the above example is that as n increases, we expect the assignment to each individual formula should decrease. Moreover, we want the distribution to individual formulae to be undertaken according to some well-understood principles. In other words, we do not want an "ad hoc" distribution. To address this need, we turn to game theory in the next section to provide a well-behaved and principled solution.

6. Games in coalitional form – Shapley value

In this section we give the definitions of games in coalitional form and of the Shapley value.

Definition 7. Let $N = \{1, \ldots, n\}$ be a set of n players. A *game in coalitional form* is given by a function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$.

This framework defines games in a very abstract way, focusing on the possible coalition formations. A coalition is just a subset of N. This function gives what payoff can be achieved by each coalition in the game v when all its members act together as a unit.

There are numerous questions that are worthwhile to investigate in this framework. One of these questions is to know how much each player can expect in a given game v. This depends on her position in the game, i.e. what she brings to different coalitions.

Often the games are super-additive.

Definition 8. A game is *super-additive* if for each $T, U \subseteq N$ with $T \cap U = \emptyset$,

$$v(T \cup U) \geqslant v(T) + v(U)$$

In super-additive games when two coalitions join, then the joined coalition wins at least as much as (the sum of) the initials coalitions. In particular, in super-additive games, the grand coalition N is the one that brings the higher utility for the society N. The problem is how this utility can be shared among the players.[2]

Example 7. Let $N = \{1, 2, 3\}$, and let v be the following coalitional game:

$$\begin{array}{ccc} v(\{1\}) = 1 & v(\{2\}) = 0 & v(\{3\}) = 1 \\ v(\{1,2\}) = 10 & v(\{1,3\}) = 4 & v(\{2,3\}) = 11 \\ & v(\{1,2,3\}) = 12 & \end{array}$$

This game is clearly super-additive. The grand coalition can bring 12 to the three players. This is the highest utility achievable by the group. But this is not the main aim for all the players. In particular one can note that two coalitions can bring nearly as much, namely $\{1, 2\}$ and $\{2, 3\}$ that gives respectively 10 and 11, that will have to be shared only between 2 players. So it is far from certain that the grand coalition will form in this case. Another remark on this game is that all the players do not share the same situation. In particular player 2 is always of a great value for any coalition she joins. So she seems to be able to expect more from this game than the other players. For example she can make an offer to player 3 for making the coalition $\{2, 3\}$, that brings 11, that will be split into 8 for player 2 and 3 for player 3. As it will be hard for player 3 to win more than that, she will certainly accept.

A solution concept has to take into account these kinds of arguments. It means that one wants to *solve* this game by stating what is the payoff that is "due" to each agent. That requires being able to quantify the payoff that an agent can

[2] One supposes the transferable utility (TU) assumption, i.e. the utility is a common unit between the players and sharable as needed (roughly, one can see this utility as some kind of money).

claim with respect to the power that her position in the game offers (for example if she always significantly improves the payoff of the coalitions she joins, if she can threaten to form another coalition, etc.).

Definition 9. A *value* is a function that assigns to each game v a vector $S(v)$ in $\mathbb{R}^n$. $S_i(v)$ stands for player i's payoff in the game.

This function gives the payoff that can be expected by each player i for the game v, i.e. it measures i's power in the game v.

Shapley proposes a beautiful solution to this problem [50]. Basically the idea can be explained as follows: considering that the coalitions form according to some order (a first player enters the coalition, then another one, then a third one, etc.), and that the payoff attached to a player is its marginal utility (i.e. the utility that it brings to the existing coalition), so if C is a coalition (subset of N) not containing i, player's i marginal utility is $v(C \cup \{i\}) - v(C)$. As one cannot make any hypothesis on which order is the correct one, suppose that each order is equally probable. This leads to the following definition of the Shapley value of a game. For this definition, let σ be a permutation on N, with σ_n denoting all the possible permutations on N. Let us note

$$p_\sigma^i = \big\{ j \in N \bigm| \sigma(j) < \sigma(i) \big\}$$

That means that p_σ^i represents all the players that precede player i for a given order σ.

Definition 10. The *Shapley value* of a game v is defined as:

$$S_i(v) = \frac{1}{n!} \sum_{\sigma \in \sigma_n} v\big(p_\sigma^i \cup \{i\}\big) - v\big(p_\sigma^i\big)$$

The Shapley value can be directly computed from the possible coalitions (without looking at the permutations), with the following expression [50]:

$$S_i(v) = \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!} \big(v(C) - v\big(C \setminus \{i\}\big)\big)$$

where c is the cardinality of C.

Example 8. The Shapley value of the game defined in Example 7 is $(\frac{17}{6}, \frac{35}{6}, \frac{20}{6})$.

These values show that it is player 2 that is the best placed in this game, according to what we explained when we presented Example 7.

Besides this value, Shapley proposes axiomatic properties a value should have.

- $\sum_{i \in N} S_i(v) = v(N)$ **(Efficiency)**
- If i and j are such that for all C s.t. $i, j \notin C$, $v(C \cup \{i\}) = v(C \cup \{j\})$, then $S_i(v) = S_j(v)$ **(Symmetry)**
- If i is such that $\forall C\ v(C \cup \{i\}) = v(C)$, then $S_i(v) = 0$ **(Dummy)**
- $S_i(v + w) = S_i(v) + S_i(w)$ **(Additivity)**

These four axioms seem quite sensible. Efficiency states that the payoff available to the grand coalition N must be efficiently redistributed to the players (otherwise some players could expect more than what they have). Symmetry ensures that it is the role of the player in the game in coalitional form that determines her payoff, so it is not possible to distinguish players by their name (as far as payoffs are concerned), but only by their respective merits/possibilities. So if two players always are identical for the game, i.e. if they bring the same utility to every coalition, then they have the same value. The dummy player axiom says simply that if a player is of no use for any coalition, this player does not deserve any payoff. And additivity states that when we join two different games v and w in a whole super-game $v + w$ ($v + w$ is straightforwardly defined as the function that is the sum of the two functions v and w, that means that each coalition receives as payoff in the game $v + w$ the payoff it has in v plus the payoff it has in w), then the value of each player in the supergame is simply the sum of the values in the compound games.

These properties look quite natural, and the nice result shown by Shapley is that they characterize exactly the value he defined [50]:

Proposition 2. *The Shapley value is the only value that satisfies all of Efficiency, Symmetry, Dummy and Additivity.*

This result supports several variations: there are other equivalent axiomatizations of the Shapley value, and there are some different values that can be defined by relaxing some of the above axioms (see [2]).

7. Inconsistency values using Shapley value

Given an inconsistency measure, the idea is to take it as the payoff function defining a game in coalitional form, and then using the Shapley value to compute the part of the inconsistency that can be imputed to each formula of the belief base.

This allows us to combine the power of inconsistency measures based on variables and hence discriminating between singleton inconsistent belief base (like the test action values of [32], or the Inc measure in [18]), and the use of the Shapley value for knowing what is the responsibility of a given formula in the inconsistency of the belief base.

We just require some basic properties on the underlying inconsistency measure.

Definition 11. An inconsistency measure I is called a *basic inconsistency measure* if it satisfies the following properties, $\forall K, K' \in \mathcal{K}_{\mathcal{L}}$, $\forall \alpha, \beta \in \mathcal{L}$:

- $I(K) = 0$ iff K is consistent **(Consistency)**
- $I(K \cup K') \geqslant I(K)$ **(Monotony)**
- If α is a free formula of K, then $I(K) = I(K \setminus \{\alpha\})$ **(Free formula independence)**
- If $\alpha \vdash \beta$ and $\alpha \nvdash \bot$, then $I(K \cup \{\alpha\}) \geqslant I(K \cup \{\beta\})$ **(Dominance)**

We ask for few properties on the underlying inconsistency measure. The Consistency property states that a consistent base has a null inconsistency measure. The Monotony property says that the amount of inconsistency of a belief base can only grow if one adds new formulae (defined on the same language). The free formula independence property states that adding a formula that does not cause any inconsistency cannot change the inconsistency measure of the base. The Dominance property states that logically stronger formulae bring (potentially) more conflicts. Note, for the Dominance property, the condition that α is consistent ensures that β is not trivially implied by α and therefore not trivially adding more inconsistency to Δ than α.

We could also ask for the following Normalization property of the inconsistency measure, as it can be used for simplification purposes. However, we do not regard it as mandatory.

- $0 \leqslant I(K) \leqslant 1$ **(Normalization)**

Now we are able to define the Shapley Inconsistency Value.

Definition 12. Let I be a basic inconsistency measure. We define the corresponding *Shapley Inconsistency Value* (SIV), noted S^I, as the Shapley value of the coalitional game defined by the function I, i.e. let $\alpha \in K$:

$$S^I_\alpha(K) = \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \big(I(C) - I\big(C \setminus \{\alpha\}\big)\big)$$

where n is the cardinality of K and c is the cardinality of C.

Note that this SIV gives a value for each formula of the base K, so if one considers the base K as the vector $K = (\alpha_1, \ldots, \alpha_n)$, then we will use $S^I(K)$ to denote the vector of corresponding SIVs, i.e.

$$S^I(K) = \big(S^I_{\alpha_1}(K), \ldots, S^I_{\alpha_n}(K)\big)$$

This definition allows us to define to what extent a formula inside a belief base is concerned with the inconsistencies of the base. It allows us to draw a precise picture of the contradictions occurring in the base.

From this value, one can define an inconsistency value for the whole belief base as in the next definition which essentially says that a base is as bad as its worst element.

Definition 13. Let K be a belief base, $\hat{S}^I(K) = \max_{\alpha \in K} S^I_\alpha(K)$.

One can identify other aggregation functions to define the inconsistency measure of the belief base from the inconsistency measure of its formulae, such as the leximax for instance. Leximax is a refinement of max, and roughly it consists in ordering the set of values in a decreasing order, and to compare the ordered lists lexicographically. See [33] for instance for a formal definition. Taking the maximum will be sufficient for us to have valuable results and to compare this with the existing measures from the literature. Note that taking the sum as the aggregation function is not a good choice here, since as shown by the Distribution property of Proposition 6 this equals $I(K)$, "erasing" the use of the Shapley value.

We think that the most interesting measure is S^I, since it describes more accurately the inconsistency of the base. But we define $\hat{S}^I$ since it is a more concise measure, and since it is of the same type as existing ones (it associates a real to each base), that is convenient to compare our framework with existing measures.

Let us now see three instantiations of SIVs.

7.1. Drastic Shapley Inconsistency Value

We will start this section with the simplest inconsistency measure one can define, that is the drastic inconsistency value of Definition 5:

$$I_d(K) = \begin{cases} 0 & \text{if } K \text{ is consistent} \\ 1 & \text{otherwise} \end{cases}$$

This measure is not of great interest by itself, since it corresponds to the usual dichotomy of classical logic. But it will be useful to illustrate the use of the Shapley Inconsistency Values, since, even with this over-simple measure, one will produce interesting results.

Proposition 3. *The drastic inconsistency value is a basic inconsistency measure.*

Let us now illustrate the behaviour of this value on some examples.

Example 9. $K_7 = \{a, \neg a, b\}$. As b is a free formula, it has a value of 0, the two other formulae are equally responsible for the inconsistency. Then $I_d(\{a, \neg a\}) = I_d(\{a, \neg a, b\}) = 1$, and the value is $S^{I_d}(K_7) = (\frac{1}{2}, \frac{1}{2}, 0)$. So $\hat{S}^{I_d}(K_7) = \frac{1}{2}$.

Example 10. $K_8 = \{a, b, b \wedge c, \neg b \wedge d\}$. The last three formulae are the ones that belong to some inconsistency, and the last one is the one that causes the most problems (indeed removing only this formula restores the consistency of the base). As a result, the value is $S^{I_d}(K_8) = (0, \frac{1}{6}, \frac{1}{6}, \frac{4}{6})$. And $\hat{S}^{I_d}(K_8) = \frac{2}{3}$.

Example 11. $K_5 = \{a \wedge \neg a, b, \neg b, c\}$. The value is $S^{I_d}(K_5) = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6}, 0)$. So $\hat{S}^{I_d}(K_5) = \frac{2}{3}$.

Contradictory formulae (like $a \wedge \neg a$) are the most problematical ones, but they are not the only source of conflict in the base. This is exactly what is expressed in the values obtained in the above example.

7.2. MI Shapley Inconsistency Value

We now consider another syntactic inconsistency measure which, in a sense, is sensitive to the "number of conflicts" in the base. That is the MI inconsistency measure defined in Definition 6:

$$I_{MI}(K) = \left|\mathsf{MI}(K)\right|$$

So this measure evaluates the amount of conflict of the base as the number of minimal inconsistent subsets of this base, so it computes in a sense the number of different conflict points in the base.

It is easy to check that:

Proposition 4. *The MI inconsistency measure is a basic inconsistency measure.*

Let us illustrate the behaviour of this value on some examples.

Example 12. $K_7 = \{a, \neg a, b\}$. As b is a free formula, it has a value of 0, the two other formulae are equally responsible for the inconsistency. Then $I_{MI}(\{a, \neg a\}) = I_{MI}(\{a, \neg a, b\}) = 1$, and the value is $S^{I_{MI}}(K_7) = (\frac{1}{2}, \frac{1}{2}, 0)$. So $\hat{S}^{I_{MI}}(K_7) = \frac{1}{2}$.

Example 13. $K_8 = \{a, b, b \wedge c, \neg b \wedge d\}$. The last three formulae are the ones that belong to some inconsistency, and the last one is the one that causes the most problems (removing only this formula restores the consistency). Then the value is $S^{I_{MI}}(K_8) = (0, \frac{1}{2}, \frac{1}{2}, 1)$. So $\hat{S}^{I_{MI}}(K_8) = 1$.

Example 14. $K_5 = \{a \wedge \neg a, b, \neg b, c\}$. The value is $S^{I_{MI}}(K_5) = (1, \frac{1}{2}, \frac{1}{2}, 0)$. So $\hat{S}^{I_{MI}}(K_5) = 1$.

Example 15. $K_9 = \{a \wedge \neg a, b, \neg b \wedge c, \neg b \wedge d\}$. The value is $S^{I_{MI}}(K_9) = (1, 1, \frac{1}{2}, \frac{1}{2})$. So $\hat{S}^{I_{MI}}(K_9) = 1$.

The last two examples above show that this measure is very sensitive to contradictory formulae, since as soon as the base contains a contradictory formula, the inconsistency measure of the base is maximum.

Example 16. $K_{10} = \{a, \neg a \wedge b, \neg a \wedge c, \neg a \wedge d\}$. The value is $S^{I_{MI}}(K_{10}) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. So $\hat{S}^{I_{MI}}(K_{10}) = \frac{3}{2}$.

Example 17. $K_{11} = \{a, \neg a \wedge b, \neg a \vee c, \neg c\}$. The value is $S^{I_{MI}}(K_{11}) = (\frac{5}{6}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3})$. So $\hat{S}^{I_{MI}}(K_{11}) = \frac{5}{6}$.

7.3. LP_m Shapley Inconsistency Value

Let us turn now to a value that use the I_{LP_m} inconsistency measure (defined Section 4).

Unfortunately the I_{LP_m} inconsistency measure is not a basic inconsistency measure, since it does not satisfy the **free formula independence** property. Let us show this on an example.

Example 18. Consider $K_{16} = \{((a \wedge \neg b) \vee (b \wedge a \wedge \neg a)) \wedge c \wedge \neg c\}$ and $\alpha = b$. α is a free formula of $K_{16} \cup \{\alpha\}$ since the unique formula of K_{16} is already a contradiction, but it increases the inconsistency value since $I_{LP_m}(K_{16}) = \frac{1}{3}$, whereas $I_{LP_m}(K_{16} \cup \{\alpha\}) = \frac{2}{3}$.

To not satisfy **free formula independence** is problematic since this property expresses a kind of independence between the amount of information and the amount of contradiction of a base. The aim is to ensure that adding new formulas that do not enter into any contradiction/conflict do not change the inconsistency measure.

On the other hand, one argument against **free formula independence** is that it considers the contradiction/conflict only at the level of the subsets (since being a free formula means that it does not introduce any new minimal inconsistent subset). What the previous example shows is that a formula that does not induce any minimal inconsistent subset can still increase the conflicts in existing minimal inconsistent subsets.

So, in some cases, this property can be considered as too strong. To address this, we can define a weaker family of inconsistency measures.

Definition 14. For $\alpha \in K$, α is a *safe formula* in K iff $\mathsf{Atoms}(\alpha) \cap \mathsf{Atoms}(K) = \emptyset$ and $\alpha \nvdash \perp$.

Using this definition, we can introduce the following property of safe formula independence (also called weak independence by Thimm [52]).

- If α is a safe formula of K, then $I(K) = I(K \setminus \{\alpha\})$ **(Safe formula independence)**

Obviously **safe formula independence** is a (logically) weaker notion than **free formula independence**. The idea is similar, meaning that if we add new pieces of information that have no relation with the existing conflicts of the base, then the inconsistency measure does not change.

Definition 15. An inconsistency measure I is called a *weak inconsistency measure* if it satisfies **consistency**, **monotony**, **dominance**, and **safe formula independence**.

So now we can show that:

Proposition 5. *The I_{LP_m} inconsistency measure is a weak inconsistency measure.*

Now let us see the behaviour of this value on some examples.

Example 19. Let $K_5 = \{a \wedge \neg a, b, \neg b, c\}$ and $K'_5 = \{a \wedge \neg a \wedge b \wedge \neg b \wedge c\}$.
Then $S^{I_{LPm}}(K_5) = (\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, 0)$, and $\hat{S}^{I_{LPm}}(K_5) = \frac{1}{3}$.
Whereas $S^{I_{LPm}}(K'_5) = (\frac{2}{3})$ and $\hat{S}^{I_{LPm}}(K'_5) = \frac{2}{3}$.

As we can see on this example, the SIV value allows us to make a distinction between K_5 and K'_5, since $\hat{S}^{I_{LPm}}(K'_5) = \frac{2}{3}$ whereas $\hat{S}^{I_{LPm}}(K_5) = \frac{1}{3}$. This illustrates the fact that the inconsistency is more distributed in K_5 than in K'_5. This distinction is not possible with the original I_{LP_m} value. Note that with Knight's coherence value one cannot distinguish the two bases either, since the two bases have the worst inconsistency value (maximally 0-consistent).

So this example illustrates the improvement brought by this work, compared to inconsistency measures on formulae and to inconsistency measures on variables, since neither of them was able to make a distinction between K_5 and K'_5, whereas for $\hat{S}^{I_{LPm}}$ K_5 is more consistent than K'_5.

Let us see a more striking example.

Example 20. Let $K_{12} = \{a, b, b \wedge c, \neg b \wedge \neg c\}$.
Then $S^{I_{LPm}}(K_{12}) = (0, \frac{1}{18}, \frac{4}{18}, \frac{7}{18})$, and $\hat{S}^{I_{LPm}}(K_{12}) = \frac{7}{18}$.

In this example one can easily see that it is the last formula that is the most problematic, and that $b \wedge c$ brings more conflict than b alone, which is perfectly expressed in the obtained values.

8. Logical properties of SIV

Let us see now some properties of the defined values.

Proposition 6. *Assume a given basic inconsistency measure I. Every Shapley Inconsistency Value S^I satisfies*:

- $\sum_{\alpha \in K} S^I_\alpha(K) = I(K)$ **(Distribution)**
- *If $\alpha, \beta \in K$ are such that for all $K' \subseteq K$ s.t. $\alpha, \beta \notin K'$, $I(K' \cup \{\alpha\}) = I(K' \cup \{\beta\})$, then $S^I_\alpha(K) = S^I_\beta(K)$* **(Symmetry)**
- *If α is a free formula of K, then $S^I_\alpha(K) = 0$* **(Minimality)**
- *If $\alpha \vdash \beta$ and $\alpha \nvdash \perp$, then $S^I_\alpha(K) \geqslant S^I_\beta(K)$* **(Dominance)**

The Distribution property states that the inconsistency values of the formulae sum to the total amount of inconsistency in the base ($I(K)$). The Symmetry property ensures that only the amount of inconsistency brought by a formula matters for computing the SIV. As one would expect, a formula that is not embedded in any contradiction (i.e. does not belong to any minimal inconsistent subset) will not be blamed by the Shapley Inconsistency Values. This is what is expressed in the Minimality property. The Dominance property states that logically stronger formulae bring (potentially) more conflicts.

The first three properties are a restatement in this logical framework of the properties of the Shapley value. One can note that the Additivity axiom of the Shapley value is not translated here, since it makes little sense to add different inconsistency values that are two different views of the same subject: It is like trying to add the temperature of an object measured in degrees Fahrenheit to the temperature of the same object measured in degrees Celsius. We will elaborate more on this point in the next section.

Let us turn now to the properties of the measure on belief bases.

Proposition 7.

- *$\hat{S}^I(K) = 0$ if and only if K is consistent* **(Consistency)**
- *If α is a free formula of $K \cup \{\alpha\}$, then $\hat{S}^I(K \cup \{\alpha\}) = \hat{S}^I(K)$* **(Free formula independence)**
- *$\hat{S}^I(K) \leqslant I(K)$* **(Upper bound)**
- *$\hat{S}^I(K) = I(K) > 0$ if and only if $\exists\alpha \in K$ s.t. α is inconsistent and $\forall\beta \in K$, $\beta \neq \alpha$, β is a free formula of K* **(Isolation)**

The first two properties are the ones given in Definition 11 for the basic inconsistency measures. As one can easily note an important difference is that the Monotony property and the Dominance property do not hold for the SIVs on belief bases. It is sensible since distribution of the inconsistencies matters for SIVs. The Upper bound property shows that the use of the SIV aims at looking at the distribution of the inconsistencies of the base, so the SIV on belief bases is always less or equal to the inconsistency measure given by the underlying basic inconsistency measure. The Isolation property details the case where the two measures are equals. In this case, there is only one inconsistent formula in the whole base, and all the other formulas are jointly consistent.

Let us see, on Example 21, counter-examples to monotony and dominance for SIV on belief bases:

Example 21. Let

$$K_{13} = \{a, \neg a, \neg a \wedge b\}$$
$$K_{14} = \{a, \neg a, \neg a \wedge b, a \wedge b\}$$

and

$$K_{15} = \{a, \neg a, \neg a \wedge b, b\}$$

We then obtain:

$$S^{I_d}(K_{13}) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right), \qquad S^{I_d}(K_{14}) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \quad \text{and} \quad S^{I_d}(K_{15}) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0\right)$$

So:

$$\hat{S}^{I_d}(K_{13}) = \frac{2}{3}, \qquad \hat{S}^{I_d}(K_{14}) = \frac{1}{4}, \qquad \hat{S}^{I_d}(K_{15}) = \frac{2}{3}$$

This example shows that monotony is not satisfied by SIV on belief bases. Clearly $K_{13} \subset K_{14}$, but $\hat{S}^{I_d}(K_{13}) > \hat{S}^{I_d}(K_{14})$. This is explained by the fact that the inconsistency is more *diluted* in K_{14} than in K_{13}. In K_{13} the formula a is the one that is the most blamed for the inconsistency ($S_a^{I_d}(K_{13}) = \hat{S}^{I_d}(K_{13}) = \frac{2}{3}$), since it appears in all inconsistent sets. Whereas in K_{14} inconsistencies are equally caused by a and by $a \wedge b$, that decreases the responsibility of a, and the whole inconsistency value of the base.

For a similar reason dominance is not satisfied, we clearly have $a \wedge b \vdash b$ (and $a \wedge b \nvdash \perp$), but $\hat{S}^{I_d}(K_{14}) < \hat{S}^{I_d}(K_{15})$.

9. Logical characterization of $S^{I_{MI}}$

In this section we give a complete characterization of the Shapley Inconsistency Value based on the MI inconsistency measure ($S^{I_{MI}}$).

We first show that this measure can be alternatively defined from a function based on minimal inconsistent sets, before stating additional logical properties of this measure, and finally stating the logical characterization.

9.1. $S^{I_{MI}}$ as a minimal inconsistent set value

As minimal inconsistent sets are the places in the bases where the inconsistencies lie in, they can be a good starting point to define inconsistency values.

Consider the following inconsistency value (for more details on measures based on minimal inconsistent sets see [25]).

Definition 16. MIV_C is defined as follows:

$$MIV_C(K, \alpha) = \sum_{M \in \mathsf{MI}(K) \text{ s.t. } \alpha \in M} \frac{1}{|M|}$$

Basically for each formula belonging to a minimal inconsistent set M the formula receives a penalty (i.e. blame or responsibility) inversely proportional to its size ($\frac{1}{|M|}$). So the value associated to a formula is the sum of all these local penalties.

And in fact this method that computes the value of each formula by looking successively (and uniquely) to all minimal inconsistent subsets is an alternative definition of the SIV $S^{I_{MI}}$.

Proposition 8. $S_\alpha^{I_{MI}}(K) = MIV_C(K, \alpha)$.

This result is interesting since computing Shapley values is a computationally difficult task. And this alternative definition can give us an efficient practical way to compute the result of $S^{I_{MI}}$. Based on SAT solvers, different works have studied the problem of identifying minimal inconsistent subsets (called in these works Minimally Unsatisfiable Subformulas or MUS). Although the identification problem is computationally hard, since checking whether a set of clauses is a MUS or not is DP-complete, and checking whether a formula belongs to the set of MUSes of a base, is in Σ_2^P [16]; it seems that finding each MUS can be practically feasible [20,21].

The other interest of this result is that it is useful to state the logical characterization of $S^{I_{MI}}$.

9.2. Logical properties of I_{MI}

First let us remark that the I_{MI} basic inconsistency measure satisfies two additional properties:

Proposition 9. I_{MI} *satisfies*:

- *If* $\mathsf{MI}(K \cup K') = \mathsf{MI}(K) \oplus \mathsf{MI}(K')$, *then* $I(K \cup K') = I(K) + I(K')$ **(MinInc separability)**
- *If* $M \in \mathsf{MI}(K)$, *then* $I(M) = 1$ **(MinInc)**

The first property basically expresses the fact that the inconsistency measure depends only on the minimal inconsistent subsets, so that if we can partition the belief base in two subbases without “breaking” any minimal inconsistent subset, then the global inconsistency measure is the sum of the inconsistency measure of the two subbases.

The second property really depicts the MI inconsistency measure behaviour, expressing the fact that all minimal inconsistent subset are considered equally.

9.3. On the Additivity axiom

We wrote in Section 8 that a direct translation of the Additivity axiom of Shapley's characterization has little meaning for inconsistency values. Let us recall this axiom. Let v and w be two coalitional games:

- $S_i(v+w) = S_i(v) + S_i(w)$ **(Additivity)**

As in this work we use basic inconsistency measures instead of coalitional games, a direct translation of this property would give (let I and J be two basic inconsistency measures):

- $S_\alpha^{I+J}(K) = S_\alpha^I(K) + S_\alpha^J(K)$ **(Add1)**

But first, it is strange to add different measures of inconsistency that give different evaluation of the same situation, so the addition $S_\alpha^I(K) + S_\alpha^J(K)$ has little meaning. But also it is hard to find a definition of what could be the added measure "$I+J$". So this translation seems to lead nowhere.

Still, we can express another kind of additivity property:

- $S_\alpha^I(K \cup K') = S_\alpha^I(K) + S_\alpha^I(K')$ **(Add2)**

So this translation considers the "addition" of two different bases (the set union). But this formulation is not satisfactory because it forgets the fact that new conflicts can appear when making the union of the two bases.

So we want this property to hold only when joining two bases does not create any new inconsistencies. This leads to the following Pre-Decomposability property[3]:

- If $\mathsf{MI}(K \cup K') = \mathsf{MI}(K) \oplus \mathsf{MI}(K')$, then $S_\alpha^I(K \cup K') = S_\alpha^I(K) + S_\alpha^I(K')$ **(Pre-Decomposability)**

The condition ensures that there will be no new conflicts (minimal inconsistent sets) when the two bases are joined. When this is the case, then we ask this additivity property to hold.

This property is useful only when one can split a base into two subbases without breaking minimal inconsistent sets. This is not always possible. So we need a slightly more general property:

- If $|\mathsf{MI}(K_1 \cup \cdots \cup K_n)| = |\mathsf{MI}(K_1)| + \cdots + |\mathsf{MI}(K_n)|$, then $S_\alpha^I(K_1 \cup \cdots \cup K_n) = S_\alpha^I(K_1) + \cdots + S_\alpha^I(K_n)$ **(Decomposability)**

The Decomposability property says that if we can split the minimal inconsistent sets into several subbases, then we can apply additivity on these subbases.

It is easy to see that the Pre-Decomposability property is implied by the Decomposability property.

Note that this possibility of interaction between the subgames that is not taken into account in the usual Additivity condition, is one of the criticisms about this condition. Let us quote for instance the following paragraph from [40]:

> The last condition is not nearly so innocent as the other two. For although $v+w$ is a game composed from v and w, we cannot in general expect it to be played as if it were the two separate games. It will have its own structure which will determine a set of equilibrium outcomes which may be different from those for v and w. Therefore, one might very well argue that its *a priori* value should not necessarily be the sum of the values of the two component games. This strikes us as a flaw in the concept value, but we have no alternative to suggest.

In our framework the interaction between the bases is simply the new logical conflicts that appears when joining the bases, that allows us to say when this addition can hold, and when it is not sensible.

9.4. Logical characterization of $S^{I_{MI}}$

We can now state the logical characterization of $S^{I_{MI}}$.

Proposition 10. *An inconsistency value satisfies Distribution, Symmetry, Minimality, Decomposability and MinInc if and only if it is the MI Shapley Inconsistency Value $S_\alpha^{I_{MI}}$.*

This result means that the Shapley Inconsistency Value $S_\alpha^{I_{MI}}$ is completely characterized by five simple and intuitive axioms.

[3] The property of Pre-Decomposability was called Decomposability in [25].

Note that Dominance, although satisfied by SIV, is not required for stating this proposition.

10. Applications of the SIVs

As the measures we define allow us to associate with each formula its degree of responsibility for the inconsistency of the base, they can be potentially useful for a lot of different reasoning or deliberative tasks. They can be in particular used to guide any paraconsistent reasoning, or any repair of the base. Let us quote three such possible uses for inference and belief change operators. The first example is for paraconsistent inference. The second one is about belief revision and the last one for negotiation.

10.1. Reasoning with inconsistent beliefbases

We show in this section how we can take an inconsistent beliefbase, and use the measure of inconsistency to prioritize the formulae in the beliefbase, so that a paraconsistent consequence relation can be used with the belief.

There are some, but not many, inference relations from maximal consistent subsets where the base is not stratified. The main ones are skeptical, credulous and argumentative inference relations. See [4] for a survey. There are more possibilities when the bases are stratified, with some formulas being more important than others [11,3,14]. But these approaches need an additional kind of information: the stratification. It is not a problem when such information is available from the application. But when the only information is a non-stratified base, i.e. a set of formulas, these approaches are of no use. In order to use them we need a means to induce a stratification from the flat (non-stratified) base. One approach to induce a stratification from a flat base is based on the specificity principle used for defaults such as in rational closure [38] and in System Z [47]. But this process gives a special status to implication, that is considered as a default rule, so it is not syntax independent (a formula $a \to b$ will be treated differently than the formula $\neg a \vee b$). Another approach is to use a tuple of formulae as input (as in knowledge merging) where each formula can occur multiple times in the tuple (see for example [13,26]). For this, a merged knowledge base is obtained by taking into account the degree of support that each candidate for the merged knowledge base receives from input formulae. Furthermore, this degree of support gives a preference (or priority) over the formulae in the merged knowledge that can be used to stratify them in the output.

However, there does not appear to be proposal for stratifying a set of formulae where the input is a set of formulae of propositional logic. But Shapley Inconsistency Values can be used to define such a stratification. The idea is simple: take the set of formulae and compute their Shapley Inconsistency Values. This allows us to define a stratification from less inconsistent formulas to more inconsistent ones. Then this stratification can be used as input for one of the numerous inference relations for stratified bases [11,3,14]. This allows us to extend the usual approaches to reasoning with inconsistency in a very natural way, and to define a whole set of different inference relations for flat bases by choosing one Shapley Inconsistency Value and one (stratified) inference relation.

10.2. Iterated revision and transmutation policies

The problem of belief revision is to incorporate a new piece of information that is more reliable than (and conflicting with) the old beliefs of the agent. This problem has received a nice answer in the work of Alchourrón, Gärdenfors and Makinson [1] in the one-step case. But when one wants to iterate revision, there are numerous problems and no definitive answer has been reached in the purely qualitative case [15,17]. Using a partially quantitative framework, some proposals have given interesting results (see e.g. [54,51]). Here "partially quantitative" means that the incoming piece of information needs to be labeled by a degree of confidence denoting how strongly we believe it. The problem in this framework is to justify the use of such a degree, what does it mean exactly and where does it come from. One possibility is to use an inconsistency measure (or a composite measure computed from an information measure [39,36,32] and an inconsistency measure) to determine this degree of confidence. We can then use the partially quantitative framework to derive revision operators with a nice behaviour (w.r.t. [15,5,30]). In this setting, since the degree attached to the incoming information is not a given data, but computed directly from the information itself and the agent policy (behaviour with respect to information and contradiction, encoded by a composite measure) then the problem of the justification of the meaning of the degrees is avoided.

Another possible use of the inconsistency measures for belief revision is that they allow to define non-prioritized belief revision operators [23]. One can define several revision policies for the agent. We can for instance decide that an agent accepts a new piece of information only if it brings a lot of information and few contradictions, etc.

10.3. Negotiation

The problem of negotiation has been investigated recently under the scope of belief change tools [6–8,55,41,37,19]. The problem is to define operators that take as input belief profiles (vectors of formulae[4]) and that produce a new belief profile

[4] More exactly belief profiles are vectors of belief bases. We use this simplifying assumption just for avoiding technical details here.

that aims to be less conflicting. We call these kind of operators conciliation operators. The idea followed in [7,8,37] to define conciliation operators is to use an iterative process where at each step a set of formulae is selected. These selected formulae are logically weakened. The process stops when one reaches a consensus, i.e. a consistent belief profile.[5] Many interesting operators can be defined when one fixes the selection function (the function that selects the formulae that must be weaken at each round) and the weakening method. In [37] the selection function is based on a notion of distance. It can be sensible if such a distance is meaningful in a particular application. If not, it is only an arbitrary choice. It would then be sensible to choose instead one of the inconsistency measures we defined in this paper. So the selection function would choose the formulae with the highest inconsistency value. These formulae are clearly the more problematic ones. More generally SIVs can be used to define new belief merging methods, and guide other negotiation-like operators.

11. Conclusion

We have proposed in this paper a new framework for defining inconsistency values. The SIV values we introduce allow us to take into account the distribution of the inconsistency among the formulae of the belief base and the variables of the language. This is, as far as we know, the only definition that allows us to take both types of information into account, thus allowing to have a more precise picture of the inconsistency of a belief base. The perspectives of this work are numerous. First, as sketched in the previous section, the use of inconsistency measures, and especially the use of Shapley Inconsistency Values, can be valuable for several belief change operators, for instance for modeling of negotiation. The Shapley value is not the only solution concept for coalitional games, so an interesting question is to know if other solution concepts (for a review of other values see [44]) can be sensible as a basis for defining other inconsistency measures. But the main way of research opened by this work is to study more closely the connections between other notions of (cooperative) game theory and the logical modeling of belief change operators.

12. Proofs

Proof of Proposition 3. Clearly **consistency** is satisfied by definition.

To show **monotony** is direct. If $K \cup K'$ is consistent, then K alone is also consistent, so $I_d(K \cup K') = I_d(K) = 0$, otherwise $K \cup K'$ is not consistent, so $I_d(K \cup K') = 1$ by definition, so $I_d(K \cup K') \geqslant I_d(K)$. So in either case $I_d(K \cup K') \geqslant I_d(K)$.

To show **free formula independence** proceed by cases: If K is consistent, then $I_d(K) = I_d(K \setminus \{\alpha\}) = 0$ by definition, otherwise K is not consistent, but as by hypothesis α is a free formula of K, this implies that $K \setminus \{\alpha\}$ is not consistent. Then $I_d(K) = I_d(K \setminus \{\alpha\}) = 1$ by definition. So in either case $I_d(K) = I_d(K \setminus \{\alpha\})$.

To show **dominance**, first if $K \cup \{\beta\}$ is consistent, then $I_d(K \cup \{\beta\}) = 0$, so $I_d(K \cup \{\alpha\}) \geqslant I_d(K \cup \{\beta\})$. Now, if $K \cup \{\beta\}$ is not consistent. By hypothesis $\alpha \vdash \beta$, this implies that $Cn(K \cup \{\beta\}) \subseteq Cn(K \cup \{\alpha\})$. So $K \cup \{\alpha\}$ is not consistent too. Then $I_d(K \cup \{\alpha\}) = I_d(K \cup \{\beta\}) = 1$. So also in this case we have $I_d(K \cup \{\alpha\}) \geqslant I_d(K \cup \{\beta\})$. $\square$

Proof of Proposition 4. To show **consistency**, note that K is consistent if and only if $\mathsf{MI}(K) = \emptyset$. So K is consistent if and only if $I_{MI}(K) = |\mathsf{MI}(K)| = 0$.

To show **monotony**, note that if $M \in \mathsf{MI}(K)$, then for any K', $M \in \mathsf{MI}(K \cup K')$. So $|\mathsf{MI}(K)| \leqslant |\mathsf{MI}(K \cup K')|$. That means $I_{MI}(K) \leqslant I_{MI}(K \cup K')$.

For **free formula independence**, if α is a free formula of K, this means that α does not belong to any minimal inconsistent set of K. So $\mathsf{MI}(K) = \mathsf{MI}(K \setminus \{\alpha\})$. So $I_{MI}(K) = I_{MI}(K \setminus \{\alpha\})$.

For **dominance**, as $\alpha \vdash \beta$, then for any $M \in \mathsf{MI}(K \cup \{\beta\})$, either $\beta \notin M$ or $\beta \in M$. If $\beta \notin M$, then $M \in \mathsf{MI}(K)$, and so $M \in \mathsf{MI}(K \cup \{\alpha\})$. Otherwise $\beta \in M$, and so $M \setminus \{\beta\} \cup \{\alpha\} \in \mathsf{MI}(K \cup \{\alpha\})$. So this means that $|\mathsf{MI}(K \cup \{\alpha\})| \geqslant |\mathsf{MI}(K \cup \{\beta\})|$, or equivalently $I_{MI}(K \cup \{\alpha\}) \geqslant I_{MI}(K \cup \{\beta\})$. $\square$

Proof of Proposition 5. For **consistency** note that if K is consistent, then K has at least one (classical) model. This model is also a LP_m model of K. So $\min_{\omega \in Mod_{LP}(K)}\{|\omega!|\} = 0$ since this classical model does not map any variable to the inconsistent truth value. So $I_{LP_m}(K) = 0$. If K is not consistent, then K has no classical model. So any LP_m model of K maps at least one variable to the inconsistent truth value, so $\min_{\omega \in Mod_{LP}(K)}\{|\omega!|\} \geqslant 1$, so $I_{LP_m}(K) \neq 0$.

For **monotony**, note that by definition of LP_m models $Mod_{LP}(K \cup K') \subseteq Mod_{LP}(K)$. So $\min_{\omega \in Mod_{LP}(K \cup K')}\{|\omega!|\} \geqslant \min_{\omega \in Mod_{LP}(K)}\{|\omega!|\}$. So $I_{LP_m}(K \cup K') \geqslant I_{LP_m}(K)$.

For **dominance**, if $\alpha \vdash \beta$, this means that $Mod_{LP}(K \cup \{\alpha\}) \subseteq Mod_{LP}(K \cup \{\beta\})$. So $\min_{\omega \in Mod_{LP}(K \cup \{\alpha\})}\{|\omega!|\} \geqslant \min_{\omega \in Mod_{LP}(K \cup \{\beta\})}\{|\omega!|\}$, and $I_{LP_m}(K \cup \{\alpha\}) \geqslant I_{LP_m}(K \cup \{\beta\})$.

For **safe formula independence**, for each model, $w \in Mod_{LP}(K \setminus \{\alpha\})$ such that $|w!|$ is minimal, there is a model $w' \in Mod_{LP}(K)$ such w and w' agree on the atoms occurring in $K \setminus \{\alpha\}$ and for the atoms not occurring in $K \setminus \{\alpha\}$ (i.e. those occurring in α), the assignment by w' is in $\{\mathsf{T}, \mathsf{F}\}$ (i.e. w' does not assign B to these atoms by), and hence, $|w!| = |w'!|$. Furthermore, for each $w' \in Mod(K)$ such that $|w'!|$ is minimal, there is a model $w \in Mod(K)$ such w and w' agree on

[5] A belief profile is consistent if the conjunction of its formulae is consistent.

the atoms occurring in $K \setminus \{\alpha\}$ and for the atoms not occurring in $K \setminus \{\alpha\}$, the assignment by w' is in $\{\mathsf{T}, \mathsf{F}\}$, and hence, $|w!| = |w'!|$. Therefore, if α is a safe formula in K, then $I_{LP_m}(K) = I_{LP_m}(K \setminus \{\alpha\})$. $\square$

Proof of Proposition 6. To show **distribution**, let us recall that

$$\begin{aligned} S^I_\alpha(K) &= \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \big(I(C) - I\big(C \setminus \{\alpha\}\big)\big) \\ &= \frac{1}{n!} \sum_{\sigma \in \sigma_n} I\big(p^\alpha_\sigma \cup \{\alpha\}\big) - I\big(p^\alpha_\sigma\big) \end{aligned}$$

where σ_n is the set of possible permutations on $\bar{K}$, and $p^\alpha_\sigma = \{\beta \in K \mid \sigma(\beta) < \sigma(\alpha)\}$. Now

$$\begin{aligned} \sum_{\alpha \in K} S^I_\alpha(K) &= \sum_{\alpha \in K} \frac{1}{n!} \sum_{\sigma \in \sigma_n} I\big(p^\alpha_\sigma \cup \{\alpha\}\big) - I\big(p^\alpha_\sigma\big) \\ &= \frac{1}{n!} \sum_{\sigma \in \sigma_n} \sum_{\alpha \in K} I\big(p^\alpha_\sigma \cup \{\alpha\}\big) - I\big(p^\alpha_\sigma\big) \end{aligned}$$

Now note that we can order the elements of K accordingly to σ when computing the inside sum, and this gives:

$$\begin{aligned} &= \frac{1}{n!} \sum_{\sigma \in \sigma_n} \big[I\big(\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}\}\big) - I\big(\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n-1)}\}\big)\big] \\ &\quad + \big[I\big(\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n-1)}\}\big) - I\big(\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n-2)}\}\big)\big] \\ &\quad + \cdots + \big[I\big(\{\alpha_{\sigma(1)}\}\big) - I(\emptyset)\big] \\ &= \frac{1}{n!} \sum_{\sigma \in \sigma_n} I\big(\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}\}\big) - I(\emptyset) \\ &= \frac{1}{n!}\, n!\, I(K) \\ &= I(K) \end{aligned}$$

To show **symmetry**, assume that there are $\alpha, \beta \in K$ s.t. for all $K' \subseteq K$ s.t. $\alpha, \beta \notin K'$, $I(K' \cup \{\alpha\}) = I(K' \cup \{\beta\})$. Now by definition

$$S^I_\alpha(K) = \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \big(I(C) - I\big(C \setminus \{\alpha\}\big)\big)$$

Let us show that $S^I_\alpha(K) = S^I_\beta(K)$ by showing (by cases) that the elements of the sum are the same:

- If $\alpha \notin C$ and $\beta \notin C$, then $I(C) = I(C \setminus \{\alpha\}) = I(C \setminus \{\beta\})$, so $I(C) - I(C \setminus \{\alpha\}) = I(C) - I(C \setminus \{\beta\})$.
- If $\alpha \in C$ and $\beta \in C$, then note that by hypothesis, as $\alpha, \beta \notin C \setminus \{\alpha, \beta\}$, we deduce that $I(C \setminus \{\alpha\}) = I(C \setminus \{\beta\})$. So $I(C) - I(C \setminus \{\alpha\}) = I(C) - I(C \setminus \{\beta\})$.
- If $\alpha \in C$ and $\beta \notin C$. Then $I(C) - I(C \setminus \{\beta\}) = 0$, and let us denote $I(C) - I(C \setminus \{\alpha\}) = a$. Let us denote $C = C' \cup \{\alpha\}$ with $C' \cap \{\alpha\} = \emptyset$, and $C'' = C' \cup \{\beta\}$. Now notice that $C'' = C'' \setminus \{\alpha\}$ so $I(C'') - I(C'' \setminus \{\alpha\}) = 0$. We can deduce $I(C' \cup \{\alpha\}) = I(C' \cup \{\beta\})$ by the hypothesis, and hence $I(C) = I(C'')$. Also we can deduce $I(C \setminus \{\alpha\}) = I(C'' \setminus \{\beta\})$ by the hypothesis. Therefore, we also have $I(C'') - I(C'' \setminus \{\beta\}) = a$.

Hence there is a bijection $F : \wp(K) \mapsto \wp(K)$ such that if $\alpha \in C$ and $\beta \notin C$, then $F(C) = C \setminus \{\alpha\} \cup \{\beta\}$ otherwise $F(C) = C$. So using this bijection, we have that for all $C \subseteq K$, $I(C) - I(C \setminus \{\alpha\}) = I(F(C)) - I(F(C) \setminus \{\beta\})$. Hence, $\sum_{C \subseteq K} I(C) - I(C \setminus \{\alpha\}) = \sum_{C \subseteq K} I(F(C)) - I(F(C) \setminus \{\beta\})$. Also, since F is a bijection on $\wp(K)$, $\sum_{C \subseteq K} I(C) - I(C \setminus \{\beta\}) = \sum_{C \subseteq K} I(F(C)) - I(F(C) \setminus \{\beta\})$. Therefore, $\sum_{C \subseteq K} I(C) - I(C \setminus \{\alpha\}) = \sum_{C \subseteq K} I(C) - I(C \setminus \{\beta\})$. Hence, $S^I_\alpha(K) = S^I_\beta(K)$.

To show the **minimality** property, just note that if α is a free formula of K, then by the **free formula independence** property of the basic inconsistency measure we have that for every $C \subseteq K$, such that $\alpha \in C$, $I(C) = I(C \setminus \alpha)$, so $I(C) - I(C \setminus \alpha) = 0$. Straightforwardly if $\alpha \notin C$, $I(C) = I(C \setminus \alpha)$. So the whole expression $S^I_\alpha(K) = \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} (I(C) - I(C \setminus \{\alpha\}))$ sums to 0.

Finally, to show **dominance** we will proceed in a similar way as to show **symmetry**. Assume that $\alpha, \beta \in K$ are such that $\alpha \vdash \beta$ and $\alpha \nvdash \bot$. Then, by the **dominance** property of the underlying basic inconsistency measure, we know that for all $C \subseteq K$, $I(C \cup \{\alpha\}) \geqslant I(C \cup \{\beta\})$. Now by definition of the SIV $S^I_\alpha(K) = \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} (I(C) - I(C \setminus \{\alpha\}))$. Let us show that $S^I_\alpha(K) \geqslant S^I_\beta(K)$ by showing (by cases) that the elements of the first sum are greater or equal to the corresponding elements of the second one:

- If $\alpha \notin C$ and $\beta \notin C$, then $I(C) = I(C \setminus \{\alpha\}) = I(C \setminus \{\beta\})$, so $I(C) - I(C \setminus \{\alpha\}) \geqslant I(C) - I(C \setminus \{\beta\})$.
- If $\alpha \in C$ and $\beta \in C$, then let us define C' to be such that $C \setminus \{\alpha\} = C' \cup \{\beta\}$. So we also have $C \setminus \{\beta\} = C' \cup \{\alpha\}$. Now note that by hypothesis $I(C' \cup \{\beta\}) \leqslant I(C' \cup \{\alpha\})$, so $I(C \setminus \{\alpha\}) \leqslant I(C \setminus \{\beta\})$. Hence $I(C) - I(C \setminus \{\alpha\}) \geqslant I(C) - I(C \setminus \{\beta\})$.
- If $\alpha \in C$ and $\beta \notin C$. Then $I(C) - I(C \setminus \{\beta\}) = 0$. Let us denote $C = C' \cup \{\alpha\}$ where $C' \cap \{\alpha\} = \emptyset$, and $C'' = C' \cup \{\beta\}$. Now notice that $I(C'') - I(C'' \setminus \{\alpha\}) = 0$. So $I(C'') - I(C'' \setminus \{\alpha\}) \geqslant I(C) - I(C \setminus \{\beta\})$. Note that $I(C'' \setminus \{\beta\}) = I(C \setminus \{\alpha\}) = I(C')$. As we can deduce $I(C) \geqslant I(C'')$ by the hypothesis, we also have $I(C) - I(C \setminus \{\alpha\}) \geqslant I(C'') - I(C'' \setminus \{\beta\})$.

As with symmetry, we can then obtain a bijection that allows us to show dominance. □

Proof of Proposition 7. To prove **consistency** note that if K is consistent, then for every $C \subseteq K$, $I(C) = 0$ (this is a direct consequence of the **consistency** property of the underlying basic inconsistency measure). Then for every $\alpha \in K$, $S^I_\alpha(K) = 0$. Hence $\hat{S}^I(K) = \max_{\alpha \in K} S^I_\alpha(K) = 0$. For the only if direction, by contradiction, suppose that $\hat{S}^I(K) = 0$ and that K is not consistent. As K is not consistent, then by the **consistency** property of the underlying basic inconsistency measure $I(K) = a \neq 0$. By the **distribution** property of the SIV we know that $\sum_{\alpha \in K} S^I_\alpha(K) = a \neq 0$, then $\exists \alpha \in K$ such that $S^I_\alpha(K) > 0$, so $\hat{S}^I(K) = \max_{\alpha \in K} S^I_\alpha(K) > 0$. Contradiction.

To show the **free formula independence** property, just notice that for any formula β that is a free formula of $K \cup \{\beta\}$, it is also a free formula of every one of its subsets. It is easy to see from the definition that for any $\alpha \in K$, $S^I_\alpha(K) = S^I_\alpha(K \cup \{\beta\})$. This is easier if we consider the second form of the definition: $S^I_\alpha(K) = \frac{1}{n!} \sum_{\sigma \in \sigma_n} I(p^\alpha_\sigma \cup \{\alpha\}) - I(p^\alpha_\sigma)$ where σ_n is the set of possible permutations on $\bar{K}$. Now note that for $S^I_\alpha(K \cup \{\beta\})$, the free formula does not bring any contradiction, so it does not change the marginal contribution of any other formula. Let us call the extensions of a permutation σ on K by β, all the permutations of $K \cup \{\beta\}$ whose restriction on elements of K is identical to σ, i.e. an extension of $\sigma = (\alpha_1, \ldots, \alpha_n)$ by β is a permutation $\sigma' = (\alpha_1, \ldots, \alpha_i, \beta, \alpha_{i+1}, \ldots, \alpha_n)$. Now note that there are $n+1$ such extensions, and that if σ' is an extension of σ, $I(p^\alpha_\sigma \cup \{\alpha\}) - I(p^\alpha_\sigma) = I(p^\alpha_{\sigma'} \cup \{\alpha\}) - I(p^\alpha_{\sigma'})$. So $S^I_\alpha(K \cup \{\beta\}) = \frac{1}{(n+1)!}(n+1) \sum_{\sigma \in \sigma_n} I(p^\alpha_\sigma \cup \{\alpha\}) - I(p^\alpha_\sigma)$ $= \frac{1}{n!} \sum_{\sigma \in \sigma_n} I(p^\alpha_\sigma \cup \{\alpha\}) - I(p^\alpha_\sigma) = S^I_\alpha(K)$. Since for any $\alpha \in K$, we have $S^I_\alpha(K) = S^I_\alpha(K \cup \{\beta\})$, we can also get $\hat{S}^I_\alpha(K \cup \{\alpha\}) = \hat{S}^I(K)$.

The **upper bound** property is stated by rewriting $I(K)$ as $\sum_{\alpha \in K} S^I_\alpha(K)$ with the **distribution** property of the SIV, and by recalling the definition of $\hat{S}^I(K)$ as $\max_{\alpha \in K} S^I_\alpha(K)$. Now by noticing that for every vector $a = (a_1, \ldots, a_n)$, $\max_{a_i \in a} a_i \leqslant \sum_{a_i \in a} a_i$, we conclude $\max_{\alpha \in K} S^I_\alpha(K) \leqslant \sum_{\alpha \in K} S^I_\alpha(K)$, i.e. $\hat{S}^I(K) \leqslant I(K)$.

To show **isolation** the if direction is straightforward: As α is inconsistent, K is inconsistent, and by the **consistency** property of the underlying basic inconsistency measure we know that $I(K) > 0$. By the **free formula** property of SIV, for every free formula β of K we have $S^I_\beta(K) = 0$. As by the **distribution** property we have $\sum_{\alpha \in K} S^I_\alpha(K) = I(K)$, this means that $S^I_\alpha(K) = I(K)$, and that $\hat{S}^I(K) = \max_{\alpha \in K} S^I_\alpha(K) = S^I_\alpha(K)$. So $\hat{S}^I(K) = I(K) > 0$. For the only if direction suppose that $\hat{S}^I(K) = I(K)$, that means that $\max_{\alpha \in K} S^I_\alpha(K) = I(K)$. But, by the **distribution** property we know that $I(K) = \sum_{\alpha \in K} S^I_\alpha(K)$. So it means that $\max_{\alpha \in K} S^I_\alpha(K) = \sum_{\alpha \in K} S^I_\alpha(K) = I(K)$. There exists α such that $S^I_\alpha(K) = I(K)$ (consequence of the definition of the max), and if there exists a $\beta \neq \alpha$ such that $S^I_\beta(K) > 0$, then $\sum_{\alpha \in K} S^I_\alpha(K) > I(K)$. Contradiction. So it means that there is α such that $S^I_\alpha(K) = I(K)$ and for every $\beta \neq \alpha$, $S^I_\beta(K) = 0$. That means that every β is a free formula, and that α is inconsistent. □

Proof of Proposition 8. Let us first show the following lemma that will be useful in the proof.

Lemma 1. *If a simple game in coalitional form on a set of players $N = \{1, \ldots, n\}$ is defined by a single winning coalition $C' \subseteq N$, i.e.:*

$$v(C) = \begin{cases} 1 & \text{if } C' \subseteq C \\ 0 & \text{otherwise} \end{cases}$$

Then the corresponding Shapley value is:

$$S_i(v) = \begin{cases} 0 & \text{if } i \notin C' \\ \frac{1}{|C'|} & \text{if } i \in C' \end{cases}$$

Proof of Lemma 1. By (Dummy) we get that if $i \notin C'$, then $S_i(v) = 0$. By (Efficiency) we know that the outcome of the grand coalition N must be shared in the sum of the Shapley values of the players: $\sum_{i \in N} S_i(v) = 1$. Since for players $i \notin C'$ we know that $S_i(v) = 0$, it means that this has to be split between the members of C'. So $\sum_{i \in C'} S_i(v) = 1$. Now by (Symmetry) we get that for all $i, j \in C'$, we have $S_i(v) = S_j(v)$. So this implies that if $i \in C'$, then $S_i(v) = \frac{1}{|C'|}$.

Let us now state the result. First suppose that α is a free formula of K, then we have immediately by (Minimality) that $S^{I_{MI}}_\alpha(K) = 0$. We also have immediately by definition that $MIV_C(K, \alpha) = 0$. So the equality is satisfied in this case.

Now suppose that α is not a free formula of K. First remark that I_{MI} can be decomposed in $I_{MI}(C) = \sum_{M \in \mathsf{MI}(K)} \hat{M}(C)$, where $\hat{M}$ is the following characteristic function

$$\hat{M}(C) = \begin{cases} 1 & \text{if } M \subseteq C \\ 0 & \text{otherwise} \end{cases}$$

Let us denote by $\hat{M}(K)$ the game in coalitional form defined from K and the characteristic function $\hat{M}$.

So now let us start from the MI Shapley Inconsistency Value:

$$\begin{aligned} S_\alpha^{I_{MI}}(K) &= \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \big(I_{MI}(C) - I_{MI}\big(C \setminus \{\alpha\}\big)\big) \\ &= \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \left(\sum_{M \in \mathsf{MI}(K)} \hat{M}(C) - \sum_{M \in \mathsf{MI}(K)} \hat{M}\big(C \setminus \{\alpha\}\big) \right) \\ &= \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \left(\sum_{M \in \mathsf{MI}(K)} \big(\hat{M}(C) - \hat{M}\big(C \setminus \{\alpha\}\big)\big) \right) \\ &= \sum_{C \subseteq K} \sum_{M \in \mathsf{MI}(K)} \frac{(c-1)!(n-c)!}{n!} \big(\hat{M}(C) - \hat{M}\big(C \setminus \{\alpha\}\big)\big) \\ &= \sum_{M \in \mathsf{MI}(K)} \sum_{C \subseteq K} \frac{(c-1)!(n-c)!}{n!} \big(\hat{M}(C) - \hat{M}\big(C \setminus \{\alpha\}\big)\big) \\ &= \sum_{M \in \mathsf{MI}(K)} S_\alpha\big(\hat{M}(K)\big) \end{aligned}$$

Now note that by Lemma 1 we have $S_\alpha(\hat{M}(K)) = \frac{1}{|M|}$.

That gives $S_\alpha^{I_{MI}}(K) = \sum_{M \in \mathsf{MI}(K)} \frac{1}{|M|} = MIV_C(K, \alpha)$. □

Proof of Proposition 10. To prove that the MI Shapley Inconsistency Value satisfies the logical properties is easy. (Distribution), (Symmetry), (Minimality) are satisfied by all Shapley Inconsistency Values (Proposition 6).

So it remains to show (Decomposability) and (MinInc). (MinInc) is satisfied by definition since $I_{MI}(M) = |\mathsf{MI}(M)| = 1$ for any $M \in \mathsf{MI}(K)$.

For (Decomposability), note that the hypothesis $|\mathsf{MI}(K_1 \cup \cdots \cup K_n)| = |\mathsf{MI}(K_1)| + \cdots + |\mathsf{MI}(K_n)|$ implies that every minimal inconsistent set of $K_1 \cup \cdots \cup K_n$ is exactly in one K_i. So for each $M \in \mathsf{MI}(K_1 \cup \cdots \cup K_n)$, then $\exists i$ s.t. $M \in \mathsf{MI}(K_i)$ and $\forall j \neq i$ $M \notin \mathsf{MI}(K_j)$. So from the hypothesis we obtain

$$\sum_{M \in \mathsf{MI}(K_1 \cup \cdots \cup K_n)} \frac{1}{|M|} = \sum_{M \in \mathsf{MI}(K_1)} \frac{1}{|M|} + \cdots + \sum_{M \in \mathsf{MI}(K_n)} \frac{1}{|M|} \tag{1}$$

And from Proposition 8 we know that

$$S_\alpha^{I_{MI}}(K) = \sum_{M \in \mathsf{MI}(K) \text{ s.t. } \alpha \in M} \frac{1}{|M|}$$

That means that Eq. (1) is equivalent to

$$S_\alpha^{I_{MI}}(K_1 \cup \cdots \cup K_n) = S_\alpha^{I_{MI}}(K_1) + \cdots + S_\alpha^{I_{MI}}(K_n)$$

For the converse implication suppose that we have an inconsistency value that satisfies (Distribution), (Symmetry), (Minimality), (Decomposability) and (MinInc). We want to show that it is the MI Shapley Inconsistency Value. From the use of (Minimality) and (Decomposability) we have that

$$S_\alpha^I(K) = \sum_{M \in \mathsf{MI}(K)} S_\alpha^I(M)$$

Now for each M if $\alpha \notin M$ we have by (Minimality) that $S_\alpha^I(M) = 0$. And if $\alpha \in M$ then we have by (Distribution) $\sum_{\alpha \in M} S_\alpha^I(M) = I(M)$. And by (Symmetry) we have that $\forall \alpha, \beta \in M$, $S_\alpha^I(M) = S_\beta^I(M)$. So we obtain that

$$\forall \alpha \in M, \quad S_\alpha^I(M) = \frac{I(M)}{|M|}$$

and therefore

$$S_\alpha^I(K) = \sum_{M \in \mathsf{MI}(K) \text{ s.t. } \alpha \in M} \frac{I(M)}{|M|}$$

Now by (MinInc) we know that for all $M \in \mathsf{MI}(K)$, $I(M) = 1$. That gives

$$S_\alpha^I(K) = \sum_{M \in \mathsf{MI}(K) \text{ s.t. } \alpha \in M} \frac{1}{|M|}$$

That is the definition of MI Shapley Inconsistency Value. □

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